ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 145–159

STAR SEARCH — A DIFFERENT SHOW

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ABSTRACT

An extension of the line search problem is considered in which the number of directions in which the searcher can head from the origin is arbitrary, but finite. One problem under study is when the distribution of the particle to be found has a bounded support. Sufficient conditions are established under which an optimal policy exhausts a given direction before it proceeds to another one, and the optimal order of directions in which to search is found. Special cases and some extensions are considered. A second problem has a game theoretic flavor, in particular a conjecture of Gal [13] is partially resolved.

1. Introduction

Consider a searcher looking for a particle at some unknown location. We assume that the set of possible locations for this particle forms a star of a finite number of directions around the searcher's initial position. The case in which there are only two directions is a special case of the classical line search setting which we will review and provide references to shortly. In order to search a given direction, which is not being searched at the moment, the searcher must return to the origin (the initial position) and only then head in the new direction. Movement within the star is at a constant speed and without stopping, until the item is found (which may be never). The particle is considered found when the searcher actually reaches its location. In Section 2 we formalize our setting mathematically.

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Received April 30, 1992 and in revised form August 20, 1992

Two problems are considered. In the first (Section 3) the objective is to minimize the expected travel time until the particle is found, when it is assumed that the distribution of the location of the particle on the star has bounded support. We establish sufficient conditions under which an optimal search policy has the property that it visits every direction only once (thereby necessarily exhausting it). We find the optimal order of directions in which to search which turns out to be similar in nature to certain scheduling problems.

The second problem (Section 4) is a game theoretic version in which the hider chooses a distribution of the location for the particle and the seeker chooses a randomized search plan. In particular our aim is to partially resolve a conjecture posed by Gal [13] (p. 172).

The line search problem was initially posed (but not solved) by Bellman [7], who assumed that there is a known probability distribution of the location of the item to be found. In his 1964 doctoral dissertation Franck (later on published as [10]) studied the problem. In particular, Franck established necessary and sufficient conditions on the probability distribution, for the problem to have a solution. Beck [1] sharpened the results in [10]. In Beck [2] the problem is studied for the case where the right and left derivatives of the distribution at the origin approaches infinity, thereby making optimal search strategies start with an infinitesimal oscillation around the origin. In Beck and Newman [3], the minimax line search problem was studied for the first time. In Beck and Warren [4], a nonlinear objective measure is considered. Fristedt and Heath [11] give more general frameworks of which many of the previous results are special cases. In Beck and Beck [5], concrete distributions are considered. The uniform case is the only one for which exact results are achieved. For other cases, only numerical algorithms or qualitative statements are possible. Beck and Beck [6] generalizes and tightens [5]. For a survey on the linear search problem see Bruss and Robertson [8]. See also Rousseeuw [16]. An excellent book on the subject with many references to related problems is Gal [13]. Gal [12] seems to be the first to consider the multi-directional extension of the linear search problem, in which a game theoretic aspect is considered, generalizing the pure minimax results studied in Beck and Newman [3].

2. Preliminaries

A **star** is a finite set of directions (half-lines) emanating from a point (the **origin**).

Let $K = \{1, \dots, k\}$, where k is the number of directions. For $d > 0$ and $i \in K$, the pair (d, i), where $d > 0$ denotes the distance of the particle from the origin, will be called a **location**.

Let us define what is meant by a **search plan**. The physical description of the problem dictates that a search will be conducted as follows. The searcher chooses a direction and starts moving along in that direction for some amount of time t_1 (say). If the particle has not been discovered by time t_1 the searcher goes back to the origin picks a possibly different direction and moves along that direction for a time period t_2, etc. In the literature of line search problems ($k = 2$), the notion of a "generalized search plan" is considered in which the searcher is allowed to begin his search with an infinitesimal oscillation between the different rays with finite total travel distance. In this case there is no first step and one describes the search step lengths as a two sided infinite sequence $\dots, t_{-2}, t_{-1}, t_0, t_1, t_2, \dots$, rather than $t_1, t_2, \dots$. In the case of the star the description becomes a little more complicated in the sense that one two sided sequence is not enough. One extreme possibility is that one can oscillate among all direction, in which case one double sided sequence will do. Another extreme is that one can oscillate between two directions, then between two new directions, and so on, in which case the maximum number of sequences is need. Of course, anything in between is *a priori* allowed. Hence, it is clear that in order to describe any (reasonable) search plan, we will need at most $\lfloor k/2 \rfloor + 1$ sequences (with $\lfloor a \rfloor$ being the largest integer less than or equal to a). The first $\lfloor k/2 \rfloor$ sequences indexing possible infinitesimal oscillation sequences which have a finite sum (or sequences of zeros if no such oscillation is present) and the last indexing a sequence which has an infinite sum. More precisely let $\mathcal{N}$ be the set of positive integers, and let $\mathcal{Z}_-$ be the set of all nonpositive integers. Set

$$\mathcal{Z}^* = \left(\bigcup_{i=1}^{\lfloor k/2 \rfloor} \mathcal{Z}_- \times \{i\} \right) \cup \mathcal{N} \times \{\lfloor k/2 \rfloor + 1\} \tag{2.1}$$

and associate with this set the lexicographic ordering $(x, i) \le (y, j)$ if $i \le j$ or $i = j$ and $x \le y$. For $L = \mathcal{N}$ or $L = \mathcal{Z}^*$ denote by

Φ_L — the collection of all maps of the form $\varphi : L \to (0, \infty) \times K$,

and let

$(t_\ell, i_\ell) = \varphi(\ell)$ for a given $\varphi \in \Phi_L$ and $\ell \in L$.

We will call $\varphi \in \Phi_{\mathcal{N}}$ ($\varphi \in \Phi_{\mathcal{Z}^*}$) a **(generalized) search plan.** Denote $(\delta(t), \iota(t))$ the location of the searcher at time t.

Remark: The notation (2.1) might be slightly confusing. However it should be thought of as the maximal set of sequences needed. When not all of them are used (for example if there is one big oscillation among all the directions, or there are oscillations among sets of three or more directions) one simply sets $t_\ell = 0$ for the first few unneeded sequences, which might be all of them if there are no oscillations at all. ∎

For a given $\varphi \in \Phi_L$, let $T_\ell = \sum_{j\le\ell} t_j$, $N(t) = \sup\{\ell \in L | \ 2T_\ell < t\}$ (left continuous). Note that if $L = \mathcal{N}$, $N(t)$ is the number of returns to the origin before (not including) time t and $N(t)+1$ is the index of the current search. For this case, assuming, without loss of generality, that the searcher travels at unit speed, a little thought reveals that the distance of the searcher from the origin at time t is given by

$$\delta(t) = \begin{cases} t - 2T_{N(t)} & \text{if } t - 2T_{N(t)} \le t_{N(t)+1} , \\ 2T_{N(t)+1} - t & \text{otherwise,} \end{cases} \tag{2.2}$$

and the index of the direction that is searched at time t is

$$\iota(t) = i_{N(t)+1} \ . \tag{2.3}$$

For the case $L = \mathcal{Z}^*$, we define $N(t)+1$ as the next member of $\mathcal{Z}^*$ in the lexicographic ordering. This is well defined for all t, even at times where one oscillation ends and the next begins, in which case $N(t)$ is the index previous to the final index in the ending oscillating sequence. With this notation, the definition of $\delta(t)$ and $\iota(t)$ via (2.2) and (2.3), respectively, is the correct one and we adopt it for this case as well.

Given the location (d,i) of the particle and a search plan φ, we will denote the (possibly infinite) time until the searcher finds the particle by

$$T_\varphi(d,i) = \inf\{t | \ \delta(t) = d, \ \iota(t) = i\} \ . \tag{2.4}$$

For a Borel probability measure μ on $(0,\infty)\times K$ and a Borel measurable function $g : (0,\infty)\times K \to (0,\infty)$, we will use the following notation

$$\begin{aligned} p_i &= \mu[(0,\infty)\times\{i\}] , \\ F_i(t) &= \mu[(0,t]\times\{i\}]/p_i , \\ Eg(D,I) &= \sum_{i\in K} p_i \int_{(0,\infty)} g(y,i)dF_i(t) \ . \end{aligned} \tag{2.5}$$

3. Sufficient Conditions for Optimality of Exhaustive Search Plans

From here onward we make the assumption that the distribution of the location of the particle on the star has bounded support and no mass at the origin. In other words we only consider Borel probability measures for which $\mu[(0,B]\times K]=1$ for some $0<B<\infty$. Let $\cup_{i\in K}A_i\times\{i\}$ be the support (always a closed set) of μ and, to avoid trivialities assume that, for each $i\in K$, A_i is not empty. Denote $b_i=\max\{x|\ x\in A_i\}$, so that $0<b_i<\infty$. For A contained in the support of μ, we denote A^c the complement of A with respect to this support. Finally we denote by 1_A the indicator of A (1 if A occurs, 0 otherwise).

Definition 3.1: An **exhaustive (generalized) search plan** (ESP) is a search plan with $\varphi(n)=(b_{\pi(n)},\pi(n))$ for $n\in K$ ($n\in K\times\{\lfloor k/2\rfloor+1\}$ and $t_n=0$ for $n\notin\mathcal{N}\times\{\lfloor k/2\rfloor+1\}$), where π is some permutation on K. For $n>k$ ($n>(k,\lfloor k/2\rfloor+1)$), t_n can and will remain undefined.

The following is the main result of this section.

THEOREM 3.1: *If for every $i\in K$, $t[F_i(t)^{-1}-p_i]$ is nonincreasing (strictly decreasing) on A_i, then every (and only) ESPs with*

$$\frac{p_{\pi(1)}}{b_{\pi(1)}}\geq\cdots\geq\frac{p_{\pi(k)}}{b_{\pi(k)}} \tag{3.1}$$

minimize $ET_\varphi(D,I)$ over $\Phi_{\mathcal{Z}^\bullet}$ (and hence, $\Phi_{\mathcal{N}}$).

First note that, under the assumption of Theorem 3.1, if $0\in A_i$ then for every $0<x\in A_i$

$$\varliminf_{t\downarrow 0}\frac{t}{F_i(t)}=\lim_{\substack{t\downarrow 0\\ t\in A_i}}\frac{t}{F_i(t)}=\lim_{\substack{t\downarrow 0\\ t\in A_i}}\left(\frac{t}{F_i(t)}-p_it\right)\geq\frac{x}{F_i(x)}-p_ix>0\ , \tag{3.2}$$

hence $\varlimsup_{t\downarrow 0}F_i(t)/t<\infty$. This clearly holds if $0\notin A_i$, therefore, as in Theorem 12 of Beck [2], it suffices to consider search plans from $\Phi_{\mathcal{N}}$ rather than $\Phi_{\mathcal{Z}^\bullet}$.

The proof of Theorem 3.1 will be established with two lemmas. The first shows that ESPs are candidates for optimal search plans. The second, which is known and has wide applications in areas such as scheduling, queueing and sequential search, helps in determining which ESPs are optimal.

LEMMA 3.1: *Under the assumption of Theorem 3.1, for any $\varphi\in\Phi_{\mathcal{N}}$ that is not an ESP, there is an ESP ψ for which $ET_\varphi(D,I)\geq(>)ET_\psi(D,I)$.*

Proof: First, we observe that it is enough to restrict our attention to a subset Φ_0 of $\Phi_{\mathcal{N}}$ on which

(i) for every $n \in \mathcal{N}$; $\varphi(n) \in \bigcup_{i\in K} A_i \times \{i\}$,
(ii) for $m < n$ with $i_n = i_m$; $t_m < t_n$, and with
(iii) $ET_\varphi(D, I) < \infty$.

Any other search plan can be strictly improved upon with a search plan from Φ_0. The lemma is trivial for $k = 1$. Assume that it is true for all stars with at most $k-1$ rays and for all choices of μ. Without loss of generality assume that $i_1 = 1$. If $t_1 = b_1$, we use the induction hypothesis, and we are done. Otherwise let $m_1 = \inf\{n \mid n > 1,\ i_n = 1\}$. If $m_1 = \infty$ then $ET_\varphi(D, I) = \infty$, which we do not allow. Define the search plans φ_1 and φ_2 by

$$\tag{3.3} \begin{aligned} &\varphi_1(n) = \varphi(n+1) \quad n \in \mathcal{N}\ , \\ &\varphi_2(n) = \begin{cases} \varphi(m_1) & \text{if } n = 1\ , \\ \varphi(n) & \text{if } 1 < n < m_1\ , \\ \varphi(n+1) & \text{if } n \geq m_1\ . \end{cases} \end{aligned}$$

To compare φ_1 with φ, note that if $(D, I) \in (0, t_1] \cap A_1 \times \{1\}$ we have $T_\varphi(D, I) = D$ and $T_{\varphi_1}(D, I) = 2\sum_{i=2}^{m_1-1} t_i + D$. For $(D, I) \notin (0, t_1] \cap A_1 \times \{1\}$, $T_\varphi(D, I) = T_{\varphi_1}(D, I) + 2t_1$. Since $\mu((0, t_1] \times \{1\}) = p_1 F_1(t_1)$, simple manipulation gives

$$\tag{3.4} ET_\varphi(D, I) - ET_{\varphi_1}(D, I) = 2\left[t_1 - p_1 F_1(t_1) \sum_{i=1}^{m_1-1} t_i\right]\ .$$

If the right side of (3.4) is positive, then φ_1 performs strictly better than φ. Otherwise, assume that the right side of (3.4) is nonpositive. On $(0, t_1] \cap A_1 \times \{1\}$, $T_\varphi(D, I) = T_{\varphi_2}(D, I) = D$. On $(t_1, t_{m_1}] \cap A_1 \times \{1\}$, $T_\varphi(D, I) = 2\sum_{i=1}^{m_1-1} t_i + D$, while $T_{\varphi_2}(D, I) = D$. Finally, on $[(0, t_{m_1}] \cap A_1 \times \{1\}]^c$, $T_{\varphi_2}(D, I) \leq 2(t_{m_1} - t_1) + T_\varphi(D, I)$. This implies that

$$\tag{3.5} \begin{aligned} &ET_\varphi(D,I) - ET_{\varphi_2}(D, I) \\ &\quad \geq 2\left[p_1[F_1(t_{m_1}) - F_1(t_1)] \sum_{i=1}^{m_1-1} t_i - [1 - p_1 F_1(t_{m_1})](t_{m_1} - t_1)\right] \\ &\quad \geq 2\big[t_1 F_1(t_1)^{-1}[F_1(t_{m_1}) - F_1(t_1)] - [1 - p_1 F_1(t_{m_1})](t_{m_1} - t_1)\big] \\ &\quad = 2F_1(t_{m_1})\big[t_1(F_1(t_1)^{-1} - p_1) - t_{m_1}(F_1(t_{m_1})^{-1} - p_1)\big]\ , \end{aligned}$$

where the second inequality follows from the nonpositivity of the right side of (3.4). From the assumptions, the bottom right side of (3.5) is nonnegative (strictly positive), so that φ_2 (strictly) improves φ.

To complete the proof, observe that the above construction may be repeated again and again. There are three possibilities. Either after a finite number of iterations we will be able to use the inductive hypothesis (if the first step is the whole interval), or after a finite number of iterations the resulting search plan is an ESP, or this procedure continues indefinitely. For the first case it should be observed that the induction hypothesis is not applied to the original data, but rather to $\{F_i(\cdot)|\ i \neq j\}$ and $\{p_i/(1-p_j)|\ i \neq j\}$ for some j, for which it is easy to check that the assumptions of the theorem still hold (now in the strict sense). In the last case, let $\varphi^m(n) = (t_n^m, i_n^m)$ be the m^{th} iterate of the search plan. Without loss of generality, assume that there is a subsequence $\mathcal{N}_1$ for which $i_1^m = 1$. Clearly $t_1^m \uparrow b_1$ as $m \to \infty$ (otherwise $ET_\varphi(D,I) = \infty$). For $m \in \mathcal{N}_1$ denote by

$$\psi^m(n) = \begin{cases} (b_1, 1) & \text{if } n = 1, \\ \varphi^m(n) & \text{otherwise.} \end{cases} \tag{3.6}$$

The following is a crude inequality

$$ET_{\varphi^m}(D,I) + 2(b_1 - t_1^m) \geq ET_{\psi^m}(D,I)\ . \tag{3.7}$$

By the inductive hypothesis, there is a single ESP ψ which outperforms ψ^m for all $m \in \mathcal{N}_1$. Since $ET_{\varphi^m}(D,I)$ is a nonincreasing (strictly decreasing) sequence we finally have that

$$ET_\varphi(D,I) \geq (>) \lim_{m \to \infty} ET_{\varphi^m}(D,I) \geq ET_\psi(D,I)\ , \tag{3.8}$$

and the proof of the lemma is complete. ∎

The next lemma and others with a similar flavor arise in various scheduling (*e.g.*, minimizing flow time) and queueing applications (*e.g.*, the "$c\mu$ rule" for the M/G/1 queue with nonpreemptive priorities), as in (among many others) Conway, Maxwell and Miller [9], and also in sequential search problems, as in Kelly [14]. The simple proof is a standard interchange argument (e.g., Rau [15]).

LEMMA 3.2: *Let x_i, y_i be positive real numbers for $i = 1, \ldots, k$. Then*

$$\sum_{i=1}^{k} \sum_{j=1}^{i} x_{\pi(i)} y_{\pi(j)} \tag{3.9}$$

is maximized (minimized) with any permutation for which

$$\frac{x_{\pi(1)}}{y_{\pi(1)}} \le (\ge) \cdots \le (\ge) \frac{x_{\pi(k)}}{y_{\pi(k)}} \tag{3.10}$$

To complete the proof of Theorem 3.1, for a given ESP φ with permutation π on K, observe that

$$T_\varphi(D,I) = D + 2\sum_{i=1}^{k} t_{\pi(i)} 1_{\{\pi(I)>\pi(i)\}} \tag{3.11}$$

so that

$$\begin{aligned} ET_\varphi(D,I) &= ED + 2\sum_{i=1}^{k} b_{\pi(i)} \left(1 - \sum_{j=1}^{i} p_{\pi(j)}\right) \\ &= ED + 2\sum_{i=1}^{k} b_i - \sum_{i=1}^{k}\sum_{j=1}^{i} b_{\pi(i)} p_{\pi(j)} , \end{aligned} \tag{3.12}$$

and the result follows from Lemma 3.3. ∎

Remarks and Special Cases:

1. The difference in performances between any two ESP's depends on μ only through the endpoints of its support and the weight it gives each direction.
2. Theorem 3.1 implies that, if its assumption is satisfied strictly and $p_i/b_i \neq p_j/b_j$ for every $i \neq j$, then there is a unique optimal search plan.
3. For p_i proportional to b_i, all ESPs have the same performance and it is straight forward to show that

$$ET_\varphi(D,I) = ED + \sum_{i=1}^{k} b_i - \frac{\sum_{i=1}^{k} b_i^2}{\sum_{i=1}^{k} b_i}. \tag{3.13}$$

4. Requiring that when the particle is found it must be brought back to the origin adds ED on the right side of (3.12), hence none of the results change under this assumption.
5. Assume that the searcher has the freedom to start from an end point (b_i, i) for some i, rather than from the origin. If the particle is not on the ith ray, once the origin is reached we are back to the original problem. Therefore one needs to compare k permutations of the form

$$\pi_j(i) = \begin{cases} \pi(j) & \text{if } i = 1 , \\ \pi(i-1) & \text{if } 2 \le i \le j , \\ \pi(i) & \text{if } j < i \le k , \end{cases} \tag{3.14}$$

where π satisfies (3.1). For $p_j = b_j / \sum_{i \in K} b_i$, direct computation shows that the optimal starting point (b_j, j) is such that $b_j(\sum_{i \in K} b_i - 2(b_j - E[D|\ I = j]))$ is maximized (thereafter, according to remark 3, continuing with an arbitrary ESP). In particular if $E[D|\ I = j] = \alpha b_j$ for some $1/2 \leq \alpha \leq 1$ and all $j \in K$ then it is easy to show that it is optimal to start with (b_j, j) where b_j is maximized (*i.e.*, start with the longest interval).

6. Any distribution function $F(\cdot)$ which is increasing return to scale ($F(t)/t$ is nondecreasing) on its support clearly satisfies the assumption of Theorem 3.1. In particular any distribution that has a nondecreasing density on (a, b) for some $0 \leq a < b < \infty$ which is zero elsewhere is increasing return to scale on $(0, b]$, hence on its support $[a, b]$. A special case is when $F(\cdot)$ is the uniform distribution function over (a, b). From this and remark 3, if μ is the uniform distribution (normalized Lebesgue measure) over $\bigcup_{i \in K}(0, b_i) \times \{i\}$, then all ESPs are optimal.
7. For every point t where $F(\cdot)$ is differentiable with derivative $f(\cdot)$ it is easy to see that the assumption of theorem 3.1 is satisfied if and only if $F(t)[1 - pF(t)] \leq (<) tf(t)$. Note that on one side, if $p \leq 1/2$ we have that $x(1-px)$ is increasing on $[0, 1]$, hence $x(1-px) \leq 1-p$ for this case. On the other hand $x(1-x) \leq 1/4$, hence $x(1-px) \leq 1/(4p)$. In particular, if the support is the interval (a, b) and $F(\cdot)$ is differentiable there, a sufficient condition for the assumption to be satisfied is

$$tf(t) \geq (>) \begin{cases} 1/(4p) & \text{if } p > 1/2\ , \\ 1-p & \text{if } p \leq 1/2\ , \end{cases} \tag{3.15}$$

for all $t \in (a, b)$ where $0 < a < b$ necessarily satisfy

$$\frac{b}{a} \leq \begin{cases} \exp(4p) & \text{if } p > 1/2\ , \\ \exp(1/(1-p)) & \text{if } p \leq 1/2\ . \end{cases} \tag{3.16}$$

Another special case is when the conditional distribution over some interval (a,b) is uniform, *i.e.*,

$$F(t) = F(a) + q\frac{t-a}{b-a} \tag{3.17}$$

for $t \in (a, b)$ and $0 < q \leq 1 - F(a)$. In this case, it is easy to check that $F(\cdot)$ satisfies the assumption (in the strict sense) on the interval (a, b) if and only if

$$F(a)[1 - pF(a)] \leq \frac{qa}{b-a}\ . \tag{3.18}$$

In particular let $F(t) = \sum_{j=1}^{n} q_j F_j(t)$, where

$$
F_j(t) = \begin{cases} 0 & \text{if } 0 < t \le A_{j-1}, \\ (t - A_{j-1})/a_j & \text{if } A_{j-1} < t \le A_j, \\ 1 & \text{if } A_j \le t, \end{cases} \tag{3.19}
$$

with $0 \le A_0 < A_1 < \ldots < A_n$, $a_j = A_j - A_{j-1}$, $q_j > 0$ for all $j = 1, \ldots, n$ and $\sum_{j=1}^{n} q_j = 1$. Denoting $Q_j = q_1 + \ldots + q_j$, the assumption of Theorem 3.1 will be satisfied (strictly) if and only if

$$
\frac{Q_{j-1}}{A_{j-1}}(1 - pQ_{j-1}) \le \frac{q_j}{a_j} \tag{3.20}
$$

for every $2 \le j \le n$. In particular this condition holds if q_j/a_j is nondecreasing in j (which also follows from remark 6).

8. If t is an atom, $t(F(t)^{-1} - p)$ has a jump down at t. If A is the support of $F(\cdot)$ and $\sup\{x|\ x \in (0,t) \cap A\} = t$. It is easy to see that the assumption of Theorem 3.1 is satisfied on $(0,t] \cap A$ if and only if it is satisfied on $(0,t) \cap A$. Hence if A is an interval, it suffices to check that the assumption holds at points of continuity of $F(\cdot)$.

4. On a Conjecture of Gal

Gal [12], [13] has studied the problem in which the searcher chooses the best strategy so as to make the worst case relative travel distance minimal. More precisely the problem Gal considered is

$$
\inf_{\varphi \in \Phi_{\mathcal{Z}}} \sup_{(d,i)} \frac{T_\varphi(d,i)}{d} \tag{4.1}
$$

where $\mathcal{Z} = \mathcal{Z}_- \cup \mathcal{N}$ is the set of all integers. In this case, either $d > 0$ and one necessarily has to use search plans with no initial step (for every search plan with an initial step the supremum in (4.1) is infinite) or $d > \epsilon > 0$ in which case there are optimal search plans with an initial step. It turns out that any policy with $i_n - 1 = n - 1 \pmod k$ and $t_n = \alpha[k/(k-1)]^n$ with $\alpha > 0$ and $n \in \mathcal{Z}$, is optimal. For the case where $d > \epsilon$ the same is true but one can also choose a policy with a first step, i.e., $n \in \mathcal{N}$, with the restriction that $\alpha/\epsilon \le 1 + ([k/(k-1)]^{k-1} - 1)^{-1}$. The value of (4.1) turns out to be $1 + 2k[k/(k-1)]^{k-1}$ which is clearly strictly larger than 1; the value of the maximin (reversing the order of the inf and sup in (4.1)). It is straight forward to argue that the same ideas can be used to show that Z in (4.1) can be replace by Z^* with the results remaining unchanged. In his book ([13], page 172), Gal conjectures that if one is allowed to consider randomized strategies for both the hider and the seeker, then the value of the game is

$$v_k = 1 + 2 \min_{r>1} \frac{r^k - 1}{k(r-1)\ln r} , \tag{4.2}$$

with the optimal randomized search strategy for the searcher having $t_n = r_k^{n+kU}$ where $r_k > 1$ is the minimizing value in (4.2), and U is a random variable uniformly distributed on $(0,1)$.

Let Φ_I (I for *increasing*) be such that $t_n \le t_{n+1}$ and $i_n - 1 = n - 1 \pmod{k}$ for every n. We will prove that the conjecture is true under the hypothesis that only cyclic nondecreasing search plans, *i.e.*, search plans in Φ_I, are allowed. From the analysis below it will follow that Gal's conjecture is true provided that the following is.

CONJECTURE 4.1: *For every symmetric distribution of the particle on the star, and every $\varphi \notin \Phi_I$ there is a $\psi \in \Phi_I$ for which $ET_\varphi(D,I) \ge ET_\psi(D,I)$.*

Clearly (e.g., Theorem 3.1) conjecture 4.1 does not hold for asymmetric distributions. Also it should be noted that Conjecture 4.1 is stronger than what we need. In fact, it suffices that it holds for distributions described by (4.4) in the sequel.

To proceed, straight forward manipulations (Gal [13], page 172) shows that with the randomized search plan above the expected time to reach every location (d,i) is at most $v_k d$, so that it suffices to argue the validity of the following result.

THEOREM 4.1: *For every $\epsilon > 0$ there is a distribution μ such that for every nonrandomized search plan from Φ_I*

$$\frac{ET_\varphi(D,I)}{ED} \ge \frac{v_k - (2k-3)\epsilon}{1+\epsilon} . \tag{4.3}$$

Proof: We follow the approach in Beck and Newman [3]. Using the notations from Section 2, for a given $\epsilon > 0$ consider a measure μ with $p_i = 1/k$ for all $i \in K$ and

$$F_i(t) = \begin{cases} 0 & \text{for } 0 < t < b , \\ 1 - b/t & \text{for } b \le t < B , \\ 1 & \text{for } B \le t , \end{cases} \tag{4.4}$$

where $b = \epsilon/(1+\epsilon)$ and $B = be^{1/\epsilon}$. It is simple to check that $ED = E[D|\ I = i] = \int_0^B [1 - F_i(t)]dt = 1$ for every $i \in K$. It is also clear that it suffices to consider $\varphi \in \Phi_I$ with an initial step $t_1 > b$ and for which there is an $n \geq k$ such that $t_{n+1} = \ldots = t_{n+k} = B$ (otherwise $ET_\varphi(D, I) = \infty$). Letting $t_{1-k} = \ldots = t_0 = b$ and $x^+ = \max(x, 0)$, we have that (with $(B, B]$ being the empty set)

$$\begin{aligned} ET_\varphi(D,I) &= 1 + 2\sum_{\nu=1}^{n+k-1} t_\nu \mu \left[\bigcup_{j=1}^{k} (t_{\nu+j-k}, B] \times \{i_{\nu+j-k}\}\right] \\ &= 1 + 2\sum_{\nu=1}^{n+k-1} t_\nu \sum_{j=1}^{k-(\nu-n)^+} \frac{b}{t_{\nu+j-k}} k^{-1} \qquad (4.5)\\ &= 1 + \frac{2b}{k}\sum_{j=1}^{k}\sum_{\nu=1}^{n+k-j} \frac{t_\nu}{t_{\nu+j-k}}, \end{aligned}$$

hence (arithmetic geometric mean inequality),

$$\begin{aligned} ET_\varphi(D,I) &\geq 1 + \frac{2b}{k}\sum_{j=1}^{k}(n+k-j)\left[\prod_{\nu=1}^{n+k-j} \frac{t_\nu}{t_{\nu+j-k}}\right]^{1/(n+k-j)} \\ &= 1 + \frac{2b}{k}\sum_{j=1}^{k}(n+k-j)(B/b)^{(k-j)/(n+k-j)} \qquad (4.6)\\ &= 1 + \frac{2}{k(1+\epsilon)}\sum_{i=0}^{k-1} \epsilon(n+i)\exp\left(\frac{i}{\epsilon(n+i)}\right). \end{aligned}$$

Now for every $0 \leq \alpha \leq \beta \leq \gamma$ and $a \geq 0$,

$$\beta e^{a/\beta} = \int_0^a e^{x/\beta}dx + \beta \geq \int_0^a e^{x/\gamma}dx + \alpha = \gamma e^{a/\gamma} - (\gamma - \alpha), \qquad (4.7)$$

in particular when $a = i$, $\alpha = \epsilon n$, $\beta = \epsilon(n+i)$ and $\gamma = \epsilon(n+k-1)$. Hence, the bottom right side of (4.6) is bounded below by

$$1 + \frac{2}{k(1+\epsilon)}\sum_{i=0}^{k-1}\left[\epsilon(n+k-1)\exp\left(\frac{i}{\epsilon(n+k-1)}\right) - \epsilon(k-1)\right]. \qquad (4.8)$$

Finally define $r = \exp(1/[\epsilon(n+k-1)]) > 1$ so that (4.8) becomes

$$1 + \frac{2}{1+\epsilon}\left[\frac{r^k - 1}{k(r-1)\ln r} - \epsilon(k-1)\right] \geq \frac{v_k - \epsilon(2k-3)}{1+\epsilon}, \qquad (4.9)$$

which establishes the result. ∎

It is of interest to consider what happens as k becomes large. In this direction let us state and prove the following Lemma.

LEMMA 4.1: *Let $a = 1.59362426$ be the unique positive fixed point of the function $g(x) = 2(1 - e^{-x})$. Then*

$$\lim_{k\to\infty} k(r_k - 1) = a,$$

$$\lim_{k\to\infty} \frac{v_k}{k} = 2\frac{e^a - 1}{a^2} = \frac{2}{a(2-a)} = 3.08827731. \tag{4.10}$$

Remark: Note that the corresponding limits for the pure minimax solution are immediate and are given by 1 and $2e = 5.43656366$, respectively.

Proof: Denote by

$$f_k(r) = \frac{\sum_{i=0}^{k-1} r^i}{\sum_{i=0}^{k-1} ir^i} = \frac{r-1}{(k-1)(r-1) + \frac{k(r-1)}{r^k-1} - 1} . \tag{4.11}$$

Since by Schwartz's inequality

$$\left(\sum_{i=0}^{k-1} ir^i\right)^2 < \left(\sum_{i=0}^{k-1} r^i\right)\left(\sum_{i=0}^{k-1} i^2r^i\right) , \tag{4.12}$$

for every $r \geq 1$, it follows via differentiation that $f_k(\cdot)$ is a strictly decreasing function on $[1, \infty)$ for every $k \geq 2$. As $r^k - 1 > k(r-1)$ for every $r > 1$ it is implied that

$$\frac{1}{k-1} < f(r) < f(1) = \frac{2}{k-1} \tag{4.13}$$

for every $r > 1$. It is also simple to check that for every $r \geq 1$, $\{f_k(r) |\ k \geq 2\}$ is a strictly decreasing sequence.

Differentiating the function which is to be minimized in (4.2) gives that $r_k > 1$ is the unique solution of $\ln r = f_k(r)$ in $[1, \infty)$, hence $\{r_k |\ k \geq 2\}$ is a strictly decreasing sequence and it is straight forward to show that $r_k \to 1$ as $k \to \infty$. To study the rate at which this convergence takes place, first note that, from the previous paragraph,

$$\ln r_k^{k-1} = (k-1)f_k(r_k) < 2 , \tag{4.14}$$

so that from $r_k^{k-1} > 1 + (k-1)(r_k - 1)$ it follows that $(k-1)(r_k - 1) < e^2 - 1$ for every $k \geq 2$. Also

$$\frac{1}{k-1} < f(r_k) = \ln r_k < r_k - 1 \, , \tag{4.15}$$

hence $a_k = (k-1)(r_k - 1)$ is a sequence which is bounded between 1 and $e^2 - 1$. Taking an arbitrary convergent subsequence which converges to a, say, we have that $r_k^k \to e^a$ for the same subsequence. Hence, taking the limits on both sides of $\ln r_k^k = k f_k(r_k)$ and rearranging terms gives that necessarily a is the unique positive fixed point of the function $g(x) = 2(1 - e^{-x})$. This implies that the entire sequence converges to a. From this, the bottom limit in (4.10) is immediate. ∎

ACKNOWLEDGEMENT: I thank Jon Lee for suggesting the topology of the star in the first place as well as the humorous "Star Search" part of the title, and for numerous stimulating discussions.

References

[1] A. Beck, *On the linear search problem,* Isr. J. Math. **2** (1964), 221–228.

[2] A. Beck, *More on the linear search problem,* Isr. J. Math. **3** (1965), 61–70.

[3] A. Beck and D. J. Newman, *Yet more on the linear search problem,* Isr. J. Math. **8** (1970), 419–429.

[4] A. Beck and P. Warren, *The return of the linear search problem,* Isr. J. Math. **14** (1973), 169–183.

[5] A. Beck and M. Beck, *Son of the linear search problem,* Isr. J. Math. **48** (1984), 109–122.

[6] A. Beck and M. Beck, *The linear search problem rides again,* Isr. J. Math. **53** (1986), 365–372.

[7] R. Bellman, *Problem 63-9, An optimal search,* SIAM Rev. **5** (1963), 274.

[8] F. T. Bruss and J. B. Robertson, *A survey of the linear search problem,* Math. Sci. **13** (1988), 75–89.

[9] R. W. Conway, W. L. Maxwell and L. W. Miller, *Theory of Scheduling,* Addison-Wesley, Reading Massachusetts, 1967.

[10] W. Franck, *An optimal search problem,* SIAM Rev. **7** (1965), 503–512.

[11] B. Fristedt and D. Heath, *Searching for a particle on the real line,* Adv. Appl. Prob. **6** (1974), 79–102.

[12] S. Gal, *Minimax solutions for linear search problems,* SIAM J. Appl. Math **27** (1974), 17–30.

[13] S. Gal, *Search Games,* Academic Press, New York, 1980.

[14] F. P. Kelly, *A remark on search sequencing problems,* Math. Opns. Res. **7** (1982), 154–157.

[15] G. Rau, *Minimizing a function of permutations of n integers,* Opns. Res. **19** (1971), 237–240.

[16] P. J. Rousseeuw, *Optimal search paths for random variables,* J. Comput. Appl. Math. **9** (1983), 279–286.